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LETTER TO THE EDITOR

# Path integral formalism for *Osp* (1|2) coherent states*

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**Abstract.** A path-integral formulation in the representation of coherent states for the supergroup $Osp(1|2)$ is introduced. An expression for the transition amplitude connecting two $Osp(1|2)$ coherent states is constructed and the corresponding canonical equations of motion derived. A set of generalized Poisson brackets is introduced.

Both path integrals [1] and coherent states [2, 3] have played major roles in the study of quantum-mechanical systems, particularly for establishing the correspondence between classical and quantum physics. Coherent states for arbitrary Lie groups were constructed by Perelomov [4] and Gilmore [5] and generalized to the case of supergroups by others [6–8]. The use of coherent states to provide an alternative method of obtaining the phase-space path integral and, hence, Hamilton's equations of motion, was pioneered by Klauder and others [9]. This technique has been extended to include a formulation in terms of coherent states for $SU(2)$ [10], $SU(1,1)$ [11] and the $n$-dimensional Euclidean group [12]. The coherent-state path-integral formalism has also found its application in the theoretical study of Berry's geometrical phase [13]. In the past few years, there have been hints of physically realized supersymmetry in nuclear [14], atomic [15] and many-body quantum systems [16]. Recently, Schmitt and Mufti [17] presented the path-integral formalism of coherent states for the non-compact supergroup $Osp(1|2, R)$, which contains $Sp(2, R)$ as a subgroup.

In this letter, we wish to generalize the previous path-integral formalism of $SU(2)$ coherent states studied in [10] to the supergroup $Osp(1|2)$ [18], which contains $SU(2)$ as a subgroup. We first recall the $Osp(1|2)$ and its relevant irreducible representation, then construct the associated coherent states. Later, we present the path-integral formulation of the transition amplitude between two $Osp(1|2)$ coherent states and derive the classical equations of motion for the system.

The superalgebra $Osp(1|2)$ contains three even (bosonic) generators $Q_3$, $Q_\pm$ of $SU(2)$ and two odd (fermionic) generators $V_\pm$ with (anti)commutation relations

$$\begin{aligned} &[Q_3, Q_\pm] = \pm Q_\pm \qquad [Q_+, Q_-] = 2Q_3 \\ &[Q_3, V_\pm] = \pm\tfrac{1}{2}V_\pm \qquad [Q_\pm, V_\pm] = 0 \qquad [Q_\pm, V_\mp] = V_\pm \\ &\{V_\pm, V_\pm\} = \pm\tfrac{1}{2}Q_\pm \qquad \{V_\pm, V_\mp\} = -\tfrac{1}{2}Q_3. \end{aligned} \tag{1}$$

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There is a graded anti-involution $\star$

$$(Q_{\pm})^{\star} = Q_{\mp} \qquad Q_3{}^{\star} = Q_3 \qquad (V_{\pm})^{\star} = \pm(-1)^{\varepsilon} V_{\mp} \qquad \varepsilon = 0, 1$$
$$(XY)^{\star} = (-1)^{p(X)p(Y)} Y^{\star} X^{\star} \qquad (X^{\star})^{\star} = (-1)^{p(X)} X \tag{2}$$

where $p(X) = 0$, 1 is the parity of a homogeneous element $X \in Osp(1|2)$. We shall use $\varepsilon = 0$ later on.

An irreducible representation of this algebra contains a doublet of representations $|q, q, q_3\rangle$ and $|q, q - \frac{1}{2}, q_3\rangle$ of the even part $SU(2)$ with highest weights $q$ and $q - \frac{1}{2}$, respectively. Note that the quantum number $q = \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The $q$ representation is defined as follows

$$\begin{aligned}
&Q_3|q, q, q_3\rangle = q_3|q, q, q_3\rangle \\
&Q_3|q, q - \tfrac{1}{2}, q_3\rangle = q_3|q, q - \tfrac{1}{2}, q_3\rangle \\
&Q_{\pm}|q, q, q_3\rangle = \sqrt{(q \mp q_3)(q \pm q_3 + 1)}|q, q, q_3 \pm 1\rangle \\
&Q_{\pm}|q, q - \tfrac{1}{2}, q_3\rangle = \sqrt{(q - 1/2 \mp q_3)(q + 1/2 \pm q_3)}|q, q - \tfrac{1}{2}, q_3 \pm 1\rangle \\
&V_{\pm}|q, q, q_3\rangle = \mp\tfrac{1}{2}\sqrt{q \mp q_3}|q, q - \tfrac{1}{2}, q_3 \pm \tfrac{1}{2}\rangle \\
&V_{\pm}|q, q - \tfrac{1}{2}, q_3\rangle = -\tfrac{1}{2}\sqrt{q + 1/2 \pm q_3}|q, q, q_3 \pm \tfrac{1}{2}\rangle.
\end{aligned} \tag{3}$$

The $q$ representation is a grade-star representation with respect to a suitable positive-definite scalar product on the representation space.

For a doublet of Grassmann variables $\theta$ and $\bar{\theta}$, there exists a graded anti-involution $\star$, defined by

$$\theta^{\star} = \bar{\theta} \qquad \bar{\theta}^{\star} = -\theta \qquad (\theta\bar{\theta})^{\star} = (-1)^{p(\theta)p(\bar{\theta})}\bar{\theta}^{\star}\theta^{\star} = \theta\bar{\theta} \tag{4}$$

where $p(\theta) = p(\bar{\theta}) = 1$.

Now, we construct the $Osp(1|2)$ coherent states as

$$\begin{aligned}
|\alpha, \theta\rangle &= U(\xi, \theta)|q, q, -q\rangle \\
&\equiv e^{\xi Q_+ - \xi^* Q_-} e^{\theta V_+ - \bar{\theta} V_-}|q, q, -q\rangle \\
&= e^{\alpha Q_+} e^{\log(1+\alpha\alpha^*) Q_3} e^{-\alpha^* Q_-} e^{\theta V_+ - \bar{\theta} V_-}|q, q, -q\rangle
\end{aligned} \tag{5}$$

with

$$\xi = \frac{\theta}{2} e^{-i\phi} \qquad \alpha = \tan\frac{\theta}{2} e^{-i\phi} \qquad (0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \phi \leqslant 2\pi). \tag{6}$$

In the sense of grade-star representations, the $U(\xi, \theta)$ is a unitary operator, i.e., $UU^{\star} = U^{\star}U = 1$. In (5), the $|q, q, -q\rangle$ is one of the lowest-weight state vectors. By calculation, we can get

$$\begin{aligned}
|\alpha, \theta\rangle = N \sum_{n=0}^{2q} \left[\frac{(2q)!}{n!(2q-n)!}\right]^{1/2} \alpha^n |q, q, -q + n\rangle \\
- N(1 + \alpha\alpha^*)^{1/2} \frac{(2q)^{1/2}}{2} \theta \sum_{n=0}^{2q-1} \left[\frac{(2q-1)!}{n!(2q-1-n)!}\right]^{1/2} \alpha^n |q, q - \tfrac{1}{2}, -q + \tfrac{1}{2} + n\rangle
\end{aligned} \tag{7}$$

with

$$N = \{(1 + \alpha\alpha^*)[1 + \tfrac{1}{4}\bar{\theta}\theta]\}^{-q}. \tag{8}$$

The overlap between two $Osp(1|2)$ coherent states can be expressed as

$$\langle\alpha_1, \theta_1|\alpha_2, \theta_2\rangle = N_1 N_2 \left\{(1 + \alpha_1^*\alpha_2)\left[1 + \frac{(1 + \alpha_1\alpha_1^*)^{1/2}(1 + \alpha_2\alpha_2^*)^{1/2}}{4(1 + \alpha_1^*\alpha_2)}\bar{\theta}_1\theta_2\right]\right\}^{2q}. \tag{9}$$

We can find the measure of integration and decomposition of unity for the $Osp(1|2)$ coherent states as follows

$$\int \mathrm{d}\mu\,(\alpha, \theta)|\alpha, \theta\rangle\langle\alpha, \theta| = \sum_{n=0}^{2q} |q, q, -q + n\rangle\langle q, q, -q + n|$$
$$+ \sum_{n=0}^{2q-1} |q, q - \tfrac{1}{2}, -q + \tfrac{1}{2} + n\rangle\langle q, q - \tfrac{1}{2}, -q + \tfrac{1}{2} + n| = I \tag{10}$$

with

$$\mathrm{d}\mu\,(\alpha, \theta) = \frac{4}{\pi}\,\mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,\mathrm{d}^2\alpha\,\left\{\frac{1}{(1 + \alpha\alpha^*)^2}\left[1 - \frac{1}{4}\bar{\theta}\theta\right]\right\}. \tag{11}$$

Here, we have used the definitions of integration over Grassmann variables

$$\int \mathrm{d}\bar{\theta}\,\theta(1, \bar{\theta}, \theta) = 0 \qquad \int \mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,\theta\bar{\theta} = 1. \tag{12}$$

It is interesting to note that the $Osp(1|2)$ coherent states so constructed are 'closest to classical' in the sense of Perelomov [4].

Consider a Hamiltonian $\widehat{H}$, which is constructed from the generators of the supergroup. The propagator from the coherent state at time $t_2$ to the coherent state at time $t_1$ is given by

$$T(\alpha_1, \theta_1, t_1; \alpha_2, \theta_2, t_2) = \left\langle\alpha_1, \theta_1\left|\exp\left[-\frac{\mathrm{i}}{\hbar}\widehat{H}(t_1 - t_2)\right]\right|\alpha_2, \theta_2\right\rangle. \tag{13}$$

In principle, we should use a time-ordered exponential to allow for the Hamiltonian being time dependent, however, the modifications needed are straightforward and are omitted here. As usual, we divide $(t_1 - t_2)$ into $n$ equal time intervals $\epsilon = (t_1 - t_2)/n$ and take the limit $n \to \infty$

$$T = \lim_{n\to\infty}\left\langle\alpha_1, \theta_1\left|\left[1 - \frac{\mathrm{i}}{\hbar}\widehat{H}\epsilon\right]^n\right|\alpha_2, \theta_2\right\rangle. \tag{14}$$

Inserting the completeness relation (10) into each of the equal time intervals, we can rewrite $T$ as

$$T = \lim_{n\to\infty}\int\cdots\int\prod_k \mathrm{d}\mu\,(\alpha_k, \theta_k)\prod_k\left\langle\alpha_k, \theta_k\left|\left[1 - \frac{\mathrm{i}}{\hbar}\widehat{H}\epsilon\right]\right|\alpha_{k-1}, \theta_{k-1}\right\rangle$$
$$= \lim_{n\to\infty}\int\cdots\int\prod_k \mathrm{d}\mu\,(\alpha_k, \theta_k)\prod_k\langle\alpha_k, \theta_k|\alpha_{k-1}, \theta_{k-1}\rangle$$
$$\times\prod_k\left\{1 - \frac{\mathrm{i}\epsilon}{\hbar}\frac{\langle\alpha_k, \theta_k|\widehat{H}|\alpha_{k-1}, \theta_{k-1}\rangle}{\langle\alpha_k, \theta_k|\alpha_{k-1}, \theta_{k-1}\rangle}\right\}. \tag{15}$$

Here, the endpoints are $t_n = t_1$ and $t_0 = t_2$. First, the term in the curly bracket in (15) can be replaced by the exponential of the expectation value of the Hamiltonian in the limit $\epsilon \to 0$. Next, by calculation, the product term $\prod\langle\alpha_k, \theta_k|\alpha_{k-1}, \theta_{k-1}\rangle$ is expressed as

$$\begin{aligned}\prod_k\langle\alpha_k, \theta_k|\alpha_{k-1}, \theta_{k-1}\rangle &= \exp\sum_k \epsilon\frac{1}{\epsilon}\log\langle\alpha_k, \theta_k|\alpha_{k-1}, \theta_{k-1}\rangle \\ &= \exp\sum_k \epsilon\left\{q\left[\frac{1}{1+|\alpha_k|^2}\left(\alpha_k\frac{\Delta\alpha_k{}^*}{\epsilon} - \alpha_k{}^*\frac{\Delta\alpha_k}{\epsilon}\right)\left(1-\frac{1}{4}\bar{\theta}_k\theta_k\right)\right.\right. \\ &\quad \left.\left. + \frac{1}{4}\left(\frac{\Delta\bar{\theta}_k}{\epsilon}\theta_k - \bar{\theta}_k\frac{\Delta\theta_k}{\epsilon}\right)\right] + \mathrm{O}(\Delta^2)\right\} \\ &\to \exp\int_{t_2}^{t_1} q\left[\frac{\alpha\dot{\alpha}^* - \alpha^*\dot{\alpha}}{1+|\alpha|^2} + \frac{1}{4}(\dot{\bar{\theta}}\theta - \bar{\theta}\dot{\theta})\right]\left[1-\frac{1}{4}\bar{\theta}\theta\right]\mathrm{d}t \end{aligned} \tag{16}$$

where the dot denotes time derivative and $\Delta\alpha_k = \alpha_k - \alpha_{k-1}$, $\Delta\theta_k = \theta_k - \theta_{k-1}$; the terms containing second order in $\Delta$ have been neglected.

We obtain the formal expression for the transition amplitude

$$T = \int D\mu[\alpha(t), \theta(t)]\exp\left(\frac{\mathrm{i}}{\hbar}S\right) \tag{17}$$

$$S = \int_{t_2}^{t_1} L(\alpha(t), \alpha^*(t), \dot{\alpha}(t), \dot{\alpha}^*(t), \theta(t), \bar{\theta}(t), \dot{\theta}(t), \dot{\bar{\theta}}(t))\,\mathrm{d}t \tag{18}$$

where the 'Lagrangian' $L$ is given by

$$L = -\mathrm{i}\hbar q\left[\frac{\alpha\dot{\alpha}^* - \alpha^*\dot{\alpha}}{1+|\alpha|^2} + \frac{1}{4}(\dot{\bar{\theta}}\theta - \bar{\theta}\dot{\theta})\right]\left[1-\frac{1}{4}\bar{\theta}\theta\right] - H \tag{19}$$

$$H = H(\alpha, \alpha^*, \bar{\theta}\theta) = \langle\alpha, \theta|\widehat{H}|\alpha, \theta\rangle \tag{20}$$

which can be rewritten as

$$L = \left\langle\alpha, \theta\left|\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - \widehat{H}\right)\right|\alpha, \theta\right\rangle \tag{21}$$

with the aid of

$$\begin{aligned}\left\langle\alpha, \theta\left|\frac{\partial}{\partial t}\right|\alpha, \theta\right\rangle &= \left\langle\alpha, \theta\left|\left(\dot{\alpha}\frac{\partial}{\partial\alpha} + \dot{\alpha}^*\frac{\partial}{\partial\alpha^*} + \dot{\theta}\frac{\partial}{\partial\theta} + \dot{\bar{\theta}}\frac{\partial}{\partial\bar{\theta}}\right)\right|\alpha, \theta\right\rangle \\ &= -q\left[\frac{\alpha\dot{\alpha}^* - \alpha^*\dot{\alpha}}{1+|\alpha|^2} + \frac{1}{4}(\dot{\bar{\theta}}\theta - \bar{\theta}\dot{\theta})\right]\left[1-\frac{1}{4}\bar{\theta}\theta\right].\end{aligned} \tag{22}$$

To arrive at the classical limit, we consider the case $S \gg \hbar$. The dominant contribution to the transition amplitude then comes from the path where the variation of the action vanishes. Setting $\delta S = 0$, we get the Euler–Lagrange equations for the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\alpha}} - \frac{\partial L}{\partial\alpha} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\alpha}^*} - \frac{\partial L}{\partial\alpha^*} = 0 \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\bar{\theta}}} - \frac{\partial L}{\partial\bar{\theta}} = 0. \tag{24}$$

Substituting expression (19) for the Lagrangian into (23) and (24), we derive the equations of motion for the system

$$\dot{\alpha} = \frac{\mathrm{i}}{2\hbar q}(1 + \tfrac{1}{4}\bar{\theta}\theta)(1 + \alpha\alpha^*)^2\{\alpha, H\}_{\mathrm{B}} \tag{25}$$

$$\dot{\alpha}^* = \frac{\mathrm{i}}{2\hbar q}(1 + \tfrac{1}{4}\bar{\theta}\theta)(1 + \alpha\alpha^*)^2\{\alpha^*, H\}_{\mathrm{B}} \tag{26}$$

$$\dot{\theta} = \frac{\mathrm{i}}{\hbar q}[\tfrac{1}{4}\alpha^*(1 + \alpha\alpha^*)\{\alpha, H\}_{\mathrm{B}}\theta - \tfrac{1}{4}\alpha(1 + \alpha\alpha^*)\{\alpha^*, H\}_{\mathrm{B}}\theta - 2\{\theta, H\}_{\mathrm{F}}] \tag{27}$$

$$\dot{\bar{\theta}} = \frac{\mathrm{i}}{\hbar q}[\tfrac{1}{4}\alpha(1 + \alpha\alpha^*)\{\alpha^*, H\}_{\mathrm{B}}\bar{\theta} - \tfrac{1}{4}\alpha^*(1 + \alpha\alpha^*)\{\alpha, H\}_{\mathrm{B}}\bar{\theta} + 2\{\bar{\theta}, H\}_{\mathrm{F}}]. \tag{28}$$

Here, we have introduced the Poisson brackets for complex and Grassmann variables

$$\{A, B\}_{\mathrm{B}} = \left\{\frac{\partial A}{\partial\alpha^*}\frac{\partial B}{\partial\alpha} - \frac{\partial A}{\partial\alpha}\frac{\partial B}{\partial\alpha^*}\right\} \tag{29}$$

$$\{A, B\}_F = \left\{\frac{\partial A}{\partial\bar{\theta}}\frac{\overleftarrow{\partial} B}{\partial\theta} - \frac{\partial A}{\partial\theta}\frac{\overleftarrow{\partial} B}{\partial\bar{\theta}}\right\}. \tag{30}$$

Note that $\partial/\partial\theta$ and $\overleftarrow{\partial}/\partial\theta$ are left and right derivative, respectively.

Finally, we wish to point out that all the above results are identical to those in [10], i.e., in the case of $SU(2)$ as $\theta, \bar{\theta} \to 0$. This is just what we require.

In this letter, we have presented a path-integral formalism for the supergroup $Osp(1|2)$. The resulting equations of motion contain two Poisson brackets—one for the complex variable $\alpha$ and one for the Grassmann variable $\theta$. The form of the equations of motion follows from the fact that the coherent states were constructed from a supergroup. The existence of the boson (fermion) Poisson bracket in the equations of motion for the bosonic (fermionic) variables is a direct consequence of the supergroup structure.

In our next work, we will apply our obtained results to a physical system possessing $Osp(1|2)$ supersymmetry.

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